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To
European Research Office
2, Rheingau Allee
Frankfurt/Main, Germany

September 25, 1963

Final Technical Report

(Report No. 4)

Application of Spinor Methods to Riemannian Manifolds

Research Contract No. 91-591-EUC-2682

Period covered: September 1, 1962 - August 31, 1963

"The research reported in this document has been made possible through the support and sponsorship of the U.S. Government".

1. Abstract

Generalized Pauli and Dirac matrices are derived for arbitrary Riemannian spaces. The determination of such matrices is based on the theory of transformations to principal axes, which is developed from a new point of view and expressed by explicit formulae. The relationship between tensors and spinors is defined in a very general way, and H. Weyl's theory of covariant spinor differentiation (see H. Weyl, Elektron and Gravitation, Zeitschr. f. Phys., 54, 1929) is generalized in accordance with the general tensor-spinor relationship.

2. General Introduction

In his classical paper on the spinning electron P. A. M. Dirac asked whether it is possible to interpret the square sum of four variables x_i as the square of a linear form:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = (p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4)^2.$$

The coefficients p_i must then be quantities satisfying the relations

$$p_i^2 = 1, \quad p_ip_k + p_kp_i = 0 \ (i \neq k).$$

Four quantities of this kind define a certain non-commutative associative abstract algebra, which was introduced by W. K. Clifford as early as 1878: Am. Jour. of Math. 1, 1878, p.350. If we now consider $x_1, x_2, x_3, x_4 = ict$ as the coordinates of space-time, then

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$$

is the equation of the light-cone, the generators of which are the possible paths of light. In the restricted theory of relativity normal coordinate systems for space-time are connected with each other by arbitrary Lorentz transformations, i.e. by any real linear transformation which leaves the form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2$$

invariant and which does not interchange past and future. Lorentz transformations constitute a group, the "complete Lorentz group", and this group describes the homogeneity of the 4-dimensional world. This group consists of "positive" and "negative" transformations, i.e. transformations with determinants +1 and -1,

respectively. The former constitute the "restricted Lorentz group", from which the complete group is obtained by introducing in addition the spatial reflection

$$x_4 \rightarrow x_4, \quad x_j \rightarrow -x_j \qquad (j = 1,2,3).$$

An important mathematical fact is the following: any binary linear transformation with determinant of absolute value 1 induces a positive Lorentz transformation in the x_j. Transformations which differ only by a factor $e^{i\lambda}$ of absolute value 1 give rise to the same group element. A new aspect arises in the general theory of relativity. Einstein recognizes as the source of the gravitational forces the metrical structure of the world and considers this structure as a formal property of the world. According to this, it must be assumed that the world-points form a four-dimensional manifold, on which a measure determination is impressed by a non-degenerate quadratic differential form Q having one positive and three negative dimensions. In any coordinate system x_i $(i = 1,2,3,4)$, in Riemann's sense, let

$$Q = g_{ik}dx^i dx^k.$$

Physical laws will then be expressed by tensor relations that are invariant for arbitrary continuous transformations of the arguments x_i. Now the question arises whether the correspondence between the Lorentz group and the binary unimodular group has an analogon in the general theory of relativity. The purpose of our investigations is to answer this question. Attempts in this direction were made by several authors more than 30 years ago (Fock-Iwanenko, Weyl, Einstein, Schrödinger, Levi-Civita, and others, see our list of literature in the First Final Technical Report and at the end of this report. In all these attempts rigid local "four-legs" are used in order to transfer the classical spinor concept to Riemannian geometry. Our studies have the advantage of being without

such rigid restrictions. The most important conception we have introduced is our fundamental decomposition formula of the first year. It is the mathematical basis on which the theory elaborated in the second year is established. The content of these recent studies is comprehended in the abstract, cf. page 1.

3. Generalized Pauli matrices

Pauli's spin matrices

$$1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U(2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad U(3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

fulfil the commutation relations

$$U(j)U(k) + U(k)U(j) = 2\delta(jk) = \begin{cases} 2 \ (j = k), \\ 0 \ (j \neq k). \end{cases}$$

The indices are written as arguments because they denote neither covariance nor contravariance. The linear combinations

$$V_j = a_j(k)U(k)$$

are said to be generalized Pauli matrices if they fulfil the commutation relations

$$V_j V_k + V_k V_j = 2g_{jk},$$

where the functions g_{jk} are to be interpreted as the components of the metrical fundamental tensor of the space under consideration. The question now arises what conditions have to be satisfied by the coefficients

$a_j(k)$. It is answered easily: From

$$2g_{jk} = V_jV_k + V_kV_j =$$

$$= (a_j(l)a_k(m) + a_k(l)a_j(m))U(l)U(m) =$$

$$= a_j(l)a_k(m)(U(l)U(m) + U(m)U(l)) =$$

$$= 2a_j(l)a_k(l)$$

we see that the condition

$$a_j(l)a_k(l) = g_{jk}$$

has to be satisfied. Such metrical decompositions have been derived by several authors (Einstein, Vock-Iwanenko, Weyl, and others; cf. our list of literature in the First Final Technical Report and at the end of this report).

In addition, Pauli's matrices fulfil the following anti-communication relations:

$$U(2)U(3) - U(3)U(2) = 2iU(1),$$
$$U(3)U(1) - U(1)U(3) = 2iU(2),$$
$$U(1)U(2) - U(2)U(1) = 2iU(3).$$

What are the corresponding generalized relations ? In order to answer this question we use the matrix calculus. On introducing the notations

$$u = \begin{pmatrix} U(1) \\ U(2) \\ U(3) \end{pmatrix}, \qquad v = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix},$$

$$A = \begin{pmatrix} a_1(1) & a_1(2) & a_1(3) \\ a_2(1) & a_2(2) & a_2(3) \\ a_3(1) & a_3(2) & a_3(3) \end{pmatrix},$$

$$G = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix},$$

we may write the relations derived above in the following abbreviated form:

$$v = Au ,$$

$$AA' = G ,$$

and $$[u,u] = 2iu.$$

The relations under consideration are derived as follows:

$$\begin{aligned} [v,v] = [Au,Au] &= \det(A)A'^{-1}[u,u] = \\ &= 2i \det(A)A'^{-1}u = \\ &= 2i \det(A)A'^{-1}A^{-1}v = \\ &= 2i \det^{1/2}(G)G^{-1}v . \end{aligned}$$

Explicitly, they read

$$V_2V_3 - V_3V_2 = 2ig^{1/2}(g^{11}V_1 + g^{12}V_2 + g^{13}V_3),$$

$$V_3V_1 - V_1V_3 = 2ig^{1/2}(g^{12}V_1 + g^{22}V_2 + g^{23}V_3),$$

$$V_1V_2 - V_2V_1 = 2ig^{1/2}(g^{13}V_1 + g^{23}V_2 + g^{33}V_3),$$

where $g = \det(G)$ and g^{jk} are the components of the matrix G^{-1}, i.e. the contravariant components of the metrical fundamental tensor.

4. Generalized Pauli matrices and infinitesimal motions

As it is well-known, Pauli's spin matrices represent an operator of infinitesimal rotation. This fact is due to the relations of anticommutation. Now we shall show that our generalized Pauli matrices may also be regarded as representing an infinitesimal motion. Our proof is based on Lie's theory of continuous groups of transformations.

A motion is a transformation M satisfying the relation of automorphism

$$M'GM = G.$$

It may be represented by a matrix

$$M = (I + T)^{-1}(I - T),$$

where

$$T = G^{-1}S$$

and

$$S = \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix}$$

with arbitrary parameters s_1, s_2, s_3.
According to Lie, an infinitesimal motion is represented by the matrices

$$J_k = (\partial T'/\partial s_k)_{s=0}.$$

Explicit calculation yields

$$T = \begin{pmatrix} g^{13}s_2-g^{12}s_3 & g^{11}s_3-g^{13}s_1 & g^{12}s_1-g^{11}s_2 \\ g^{23}s_2-g^{22}s_3 & g^{12}s_3-g^{23}s_1 & g^{22}s_1-g^{12}s_2 \\ g^{33}s_2-g^{23}s_3 & g^{13}s_3-g^{33}s_1 & g^{23}s_1-g^{13}s_2 \end{pmatrix}$$

and

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ -g^{13} & -g^{23} & -g^{33} \\ g^{12} & g^{22} & g^{23} \end{pmatrix},$$

$$J_2 = \begin{pmatrix} g^{13} & g^{23} & g^{33} \\ 0 & 0 & 0 \\ -g^{11} & -g^{12} & -g^{13} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} -g^{12} & -g^{22} & -g^{23} \\ g^{11} & g^{12} & g^{13} \\ 0 & 0 & 0 \end{pmatrix}.$$

From this ternary representation the general anticommutation relations may be derived. They read:

$$J_2J_3 - J_3J_2 = g^{11}J_1 + g^{12}J_2 + g^{13}J_3,$$
$$J_3J_1 - J_1J_3 = g^{12}J_1 + g^{22}J_2 + g^{23}J_3,$$
$$J_1J_2 - J_2J_1 = g^{13}J_1 + g^{23}J_2 + g^{33}J_3.$$

These relations are in structural accordance with those for the matrices V_1, V_2, V_3.

Therefore, V_1, V_2, V_3 form a binary representation of the operator of infinitesimal motion.

5. Generalized Dirac matrices

The well-known Dirac matrices

$$U(1) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad U(2) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$U(3) = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad U(4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

fulfil the commutation relations

$$U(j)\, U(k) + U(k)U(j) = 2\delta(jk).$$

Therefore, the same procedure as in the case of the Pauli matrices will lead us to generalized Dirac matrices V_1, V_2, V_3, V_4:

$$V_j = a_j(k)U(k),$$

where the coefficients $a_j(k)$ have to satisfy the conditions

$$a_j(l)a_k(l) = g_{jk}.$$

We know from chapter 3 that the generalized matrices V_1, V_2, V_3, V_4 fulfil the commutation relations

$$V_jV_k + V_kV_j = 2g_{jk}.$$

These relations are of fundamental importance for any general spinor calculus.

6. Metrical decompositions $G = AA'$ and transformations on principal axes

From the preceding chapters we see that one of the fundamental problems of the spinor calculus in Riemannian geometry is to determine the coefficients $a_j(k)$ satisfying the condition

$$a_j(l)a_k(l) = g_{jk}$$

which we have written in the form

$$AA' = G.$$

This matrix equation is equivalent to the equation

$$B'GB = I,$$

where

$$B = A'^{-1}.$$

The transformation B consists of a transformation on principal axes and of a transformation of normalization. The main difficulty is to obtain the part transforming on principal axes. This problem has been solved theoretically a long time ago. There is, however, no practicable explicit formula for transformations on principal axes or, what is the same, for transformations of similitude. We have found such a formula for the simplest case of distinct roots of the characteristic polynomial belonging to the matrix to be transformed. The theory of this formula will be developed in extenso in the following chapter.

7. On the theory of similitude of matrices

Two matrices are said to be similar to each other if they represent one and the same homogeneous linear transformation in two coordinate systems which are equivalent to each other. This relation of similitude is a relation of equivalence due to the equivalence of the coordinate systems. Therefore, each matrix of a class of similitude represents the entire class.
Any homogeneous linear transformation or class of similar matrices may also be characterized by invariant quantities the values of which do not depend on coordinates. Such invariants are the coefficients and the roots of the characteristic polynomial of the transformation, rational the coefficients, irrational the roots, which are called the eigenvalues of the transformation. If all the eigenvalues are different from each other, then they will correspond to a special coordinate system with reference to which the transformation assumes diagonal form. This case shall be considered.

Any real or complex matrix R with n rows and n columns possesses a characteristic polynomial

$$f(x) = \det(xI - R) =$$
$$= x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = (x - b_1)(x - b_2) \dots (x - b_n).$$

The coefficients $a_1, a_2, \dots, a_n$ are the elementary symmetrical functions of the roots $b_1, b_2, \dots, b_n$:

$$a_1 = f_1(b_1, \dots, b_n) = -(b_1 + \dots + b_n),$$

$$\dots\dots\dots\dots\dots\dots$$

$$a_n = f_n(b_1, \dots, b_n) = (-1)^n\, b_1 \dots b_n.$$

B denotes the diagonal matrix with the diagonal elements $b_1, b_2, \dots, b_n$:

$$B = \mathrm{Diag}(b_1, b_2, \dots, b_n).$$

The matrix R and the matrix B have the same characteristic polynomial. Therefore,

$$f(R) = f(B) = 0$$

is valid according to Cayley and Hamilton. This identity is the basis of the property

$$RP = PB$$

of the matrix

$$P = a_{n-1}I + a_{n-2}(R + B) + a_{n-3}(R^2 + RB + B^2) + \dots$$

$$\dots + R^{n-1} + R^{n-2}B + \dots + RB^{n-2} + B^{n-1}.$$

The proof is very simple:

$$RP - PB =$$

$$= a_{n-1}(R - B) + a_{n-2}(R^2 - B^2) + \dots + R^n - B^n =$$

$$= f(R) - f(B) = 0.$$

If the matrix P is regular, then the relation of similitude

$$P^{-1}RP = B$$

between the matrices R and B will follow from the relation

$RP = PB$. In order to investigate the determinant of P, we shall consider the structure of P.

For this purpose we introduce the diagonal matrices

$$I_1 = \mathrm{Diag}(1,0,0,\ldots,0,0),$$
$$I_2 = \mathrm{Diag}(0,1,0,\ldots,0,0),$$
$$\cdots\cdots\cdots\cdots$$
$$I_n = \mathrm{Diag}(0,0,0,\ldots,0,1).$$

The "projectors" I_1, I_2, ..., I_n have the properties of complimentarity:

$$I_1 + I_2 + \ldots + I_n = I,$$

of idempotence:

$$I_1^2 = I_1, \ \ldots, \ I_n^2 = I_n,$$

and of orthogonality:

$$I_i I_k = 0 \quad \text{if} \quad i \neq k.$$

The matrices

$$J_1 = I - I_1, \ \ldots, \ J_n = I - I_n$$

are introduced in order to define the matrices

$$B_1 = b_1 J_1, \ \ldots, \ B_n = b_n J_n,$$

which are related to the matrix B by the identities

$$f_1(B_1,\ldots,B_n) = a_1 I + B,$$
$$\cdots\cdots\cdots\cdots$$
$$f_{n-1}(B_1,\ldots,B_n) = a_{n-1} I + a_{n-2} B + \ldots + B^{n-1}.$$

These identities describe the relationship between the elementary symmetrical functions of n variables and the elementary symmetrical functions of n - 1 variables. When we consider the matrix P in the form

$$P =$$

$$= R^{n-1} + R^{n-2}(a_1 I + B) + \dots + a_{n-1}I + a_{n-2}B + \dots + B^{n-1},$$

which is ordered with respect to the powers of R, then we see immediately the formula

$$P =$$

$$= R^{n-1} + R^{n-2}f_1(B_1,\dots,B_n) + \dots + f_{n-1}(B_1,\dots,B_n).$$

Now we distribute one of the matrices

$B_1, B_2, \dots, B_n$, let it be the matrix B_k, onto the remaining ones in the following way:

$$C_1 = B_1 + b_k I_1, \dots, C_{k-1} = B_{k-1} + b_k I_{k-1},$$

$$C_k = B_{k+1} + b_k I_{k+1}, \dots, C_{n-1} = B_n + b_k I_n.$$

The sum $C_1 + \dots + C_{n-1}$ is equal to the sum $B_1 + \dots + B_n$.

The same holds for the corresponding power sums of higher degree:

$$C_1^h + \dots + C_{n-1}^h =$$

$$= (B_1 + b_k I_1)^h + \dots + (B_{k-1} + b_k I_{k-1})^h +$$

$$+ (B_{k+1} + b_k I_{k+1})^h + \dots + (B_n + b_k I_n)^h =$$

$$= B_1^h + \dots + B_{k-1}^h + B_{k+1}^h + \dots + B_n^h +$$

$$+ b_k^h(I_1 + \dots + I_{k-1} + I_{k+1} + \dots + I_n) =$$

$$= B_1^h + \dots + B_n^h .$$

From this we obtain the relations

$$f_1(C_1,\dots,C_{n-1}) = f_1(B_1,\dots,B_n),$$
$$\dots\dots\dots\dots\dots\dots$$
$$f_{n-1}(C_1,\dots,C_{n-1}) = f_{n-1}(B_1,\dots,B_n).$$

Therefore, we may write the matrix P in the form

$$P = R^{n-1} + R^{n-2}f_1(C_1,\dots,C_{n-1}) + \dots + f_{n-1}(C_1,\dots,C_{n-1}).$$

Now it is easy to show that the matrix P will be singular if two eigenvalues of R are equal.
We assume that b_i is equal to b_k and that i is less than k. Then the matrix C_i will be equal to the matrix b_iI, which may be commuted with any matrix. Therefore, the matrix P contains the factor $R - C_i$ the determinant of which is equal to $(-1)^n f(b_i)$. Hence, the determinant of P vanishes.

From now we propose that all the eigenvalues of the matrix R are different from each other. This assumption, however, does not imply the regularity of P.

This fact may be seen by the example of the matrix

$$R = \mathrm{Diag}(1,-1),$$

the eigenvalues of which are the numbers 1 and -1. The two possible matrices B,

$$B = \mathrm{Diag}(1,-1) = R$$

and

$$B = \mathrm{Diag}(-1,1) = -R,$$

involve the two matrices

$$P = 2R$$

and

$$P = O.$$

The former one is regular, the latter one is singular.

The following is valid generally: There is at least one matrix b among all the possible matrices B, to which a regular matrix P is coordinated. This is the fundamental theorem of our theory. It is proven as follows.

By V we denote the set of all variations with repetition, by Q the set of all permutations of the eigenvalues $b_1, b_2, \dots, b_n$. The set Q consisting of n! permutations splits into (n-1)! classes of cyclic permutations, a certain one of which we denote by Z. The sets V(P), Q(P), and Z(P) of matrices P correspond to the sets V, Q, and Z of permutations. The matrix

$$S = f'(R) = a_{n-1}I + 2a_{n-2}R + \dots + nR^{n-1}$$

is the sum of all the matrices of a class Z(P). This relation is based on NEWTON's formulae which describe the connection between elementary symmetrical functions and power sums. The determinant of S is equal to the discriminant of the characteristic polynomial of f, disregarded the sign. Hence, det(S) is different from zero. On the other hand, it is equal to the sum of all determinants of the matrices from V(P). This sum, however, is equal to the sum of the determinants of all matrices from Q(P) because all matrices from the set difference V(P) - Q(P) are singular. Therefore, at least one matrix from Q(P) must be regular. This proves our fundamental theorem.

We are not able to say more about the matrices P if we do not know the structure of the radical field of f.

8. The general relationship between tensors and spinors

If we have a metrical decomposition

$$G = U'(\bar{H} \otimes H)U,$$

then the connection between a first-rank tensor u and a first-rank spinor v is given by

$$u = U^{-1}(\bar{v} \otimes v)$$

because this relation involves

$$u'Gu = (\bar{v}'\bar{H}\,\bar{v})(v'H\,v).$$

A transformation T of v which is automorphic with respect to H ($T'HT = H$) induces a transformation

$$S = U^{-1}(\bar{T} \otimes T)U$$

of u which is automorphic with respect to G

($S'GS = G$). For U and $V = U^{-1}$ we use the notations

$$U = \begin{pmatrix} u_1(11) & u_2(11) & u_3(11) & u_4(11) \\ u_1(12) & u_2(12) & u_3(12) & u_4(12) \\ u_1(21) & u_2(21) & u_3(21) & u_4(21) \\ u_1(22) & u_2(22) & u_3(22) & u_4(22) \end{pmatrix}$$

and

$$V = \begin{pmatrix} v^1(11) & v^1(12) & v^1(21) & v^1(22) \\ v^2(11) & v^2(12) & v^2(21) & v^2(22) \\ v^3(11) & v^3(12) & v^3(21) & v^3(22) \\ v^4(11) & v^4(12) & v^4(21) & v^4(22) \end{pmatrix}$$

The following relations of orthogonality are valid:

$$u_p(mn)v^p(rs) = \delta(mr)\delta(ns),$$

$$v^p(mn)u_q(mn) = \delta^p_q.$$

Now the relations between tensors and spinors may be written in the form

$$u^p = \bar{v}'V^p v \qquad (p = 1,2,3,4),$$

where

$$V^p = \begin{pmatrix} v^p(11) & v^p(12) \\ v^p(21) & v^p(22) \end{pmatrix}.$$

The only restriction we make is the condition of Hermitean symmetry:

$$\bar{V}'^p = V^p.$$

The metrical decomposition

$$G = U'(\bar{H}\otimes H)U$$

reads explicitly:

$$g_{pq} = u_p(mn)u_q(rs)\bar{h}(mr)h(ns).$$

From

$$g_{pq}v^p(mn)v^q(rs) = u_p(ab)u_q(cd)\bar{h}(ac)h(bd)v^p(mn)v^q(rs) =$$

$$= \delta(am)\delta(bn)\delta(cr)\delta(ds)\bar{h}(ac)h(bd) =$$

$$= \bar{h}(mr)h(ns)$$

we derive the important relation

$$g_{pq}V^pV^q = \bar{H}H.$$

Now we consider a metrical decomposition

$$G = AA' .$$

Using the notations

$$A = \begin{pmatrix} a_1(1) & a_1(2) & a_1(3) & a_1(4) \\ a_2(1) & a_2(2) & a_2(3) & a_2(4) \\ a_3(1) & a_3(2) & a_3(3) & a_3(4) \\ a_4(1) & a_4(2) & a_4(3) & a_4(4) \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} a^1(1) & a^2(1) & a^3(1) & a^4(1) \\ a^1(2) & a^2(2) & a^3(2) & a^4(2) \\ a^1(3) & a^2(3) & a^3(3) & a^4(3) \\ a^1(4) & a^2(4) & a^3(4) & a^4(4) \end{pmatrix},$$

we may write:

$$g_{pq} = a_p(j)a_q(j),$$

$$g^{pq} = a^p(j)a^q(j),$$

We say that

$$u(j) = a_p(j)u^p$$

are the components of the tensor u with respect to the "four-leg" A. The conversion is also possible

$$u^p = a^p(j)u(j).$$

Analogously we write

$$V(j) = a_p(j)V^p$$

and

$$u(j) = \bar{v}'V(j)v.$$

9. Weyl's concept of covariant spinor differentation

In his paper Elektron and Gravitation (Zeitschr. f. Phys., 54, 1929), H. Weyl inaugurated a concept of covariant spinor differentiation which forms the basis of our generalization. In this chapter we shall consider its essential properties.

When a metrical decomposition

$$g_{pq} = a_p(j)a_q(j)$$

is given, then the Christoffel quantities

$$\Gamma_{pqr} = (\partial g_{pq}/\partial x_r + \partial g_{rp}/\partial x_q - \partial g_{qr}/\partial x_p)/2$$

have the form

$$\Gamma_{pqr} = (a_p(j)\partial a_q(j)/\partial x_r + a_q(j)\partial a_p(j)/\partial x_r +$$

$$+ a_r(j)\partial a_p(j)/\partial x_q + a_p(j)\partial a_r(j)/\partial x_q -$$

$$- a_q(j)\partial a_r(j)/\partial x_p - a_r(j)\partial a_q(j)/\partial x_p)/2.$$

The quantities

$$W_{pqr} = \Gamma_{pqr} - a_p(j)\partial a_q(j)/\partial x_r =$$

$$= (a_p(j)\partial a_r(j)/\partial x_q - a_q(j)\partial a_r(j)/\partial x_p)/2 +$$

$$+ (a_q(j)\partial a_p(j)/\partial x_r - a_p(j)\partial a_q(j)/\partial x_r)/2 -$$

$$- a_r(j)(\partial a_q(j)/\partial x_p - \partial a_p(j)/\partial x_q)/2$$

have the property of antisymmetry:

$$W_{pqr} = - W_{qpr}.$$

They are the coefficients of an infinitesimal rotation which has the same absolute character as Levi-Civita's infinitesimal parallel displacement. The quantities

$$W_r(jk) = a^p(j)a^q(k)W_{pqr}$$

are its components with respect to the four-leg A.

Weyl assumes that the connection between tensors u and spinors v is characterized by the relations

$$u(j) = \bar{v}'U(j)v,$$

where U(1), U(2), U(3) are Pauli's spin matrices, and U(4)/i is the binary unit matrix. An infinitesimal W-rotation has the effect

$$\delta u(j) = \delta W(jk)u(k) = W_r(jk)u(k)dx^r.$$

What is its effect on the associated spinor v ?
Without any loss of generality we may assume that

$$\delta v = \delta T v = T_r v \, dx^r.$$

Now we have

$$\delta u(j) = 2\bar{v}'U(j)\delta v = 2\bar{v}'U(j)\delta T v$$

$$= \delta W(jk)u(k) = \bar{v}'\delta W(jk)U(k)v,$$

and therefore,

$$2U(j)\delta T = \delta W(jk)U(k),$$

$$4\delta T = \delta W(jk)U(j)U(k).$$

Finally, covariant spinor differentiation is defined by

$$Dv = dv + \delta v = (d + \delta T)v,$$

$$D_r v = (\partial/\partial x_r + T_r)v,$$

where

$$T_r = W_r(jk)U(j)U(k)/4.$$

10. The general concept of covariant spinor differentiation

We assume that the relationship between tensors and spinors is given in the same general way as in chapter 8. Then we have:

$$u(j) = \bar{v}'V(j)v, \qquad \bar{V}'(j) = V(j).$$

An infinitesimal rotation δ has the effect

$$\delta u(j) = \bar{v}'\ dV(j)v + 2\ \bar{v}'V(j)\delta v =$$

$$= \bar{v}'(dV(j) + 2V(j)\delta T)v =$$

$$= \bar{v}'(\partial V(j)/\partial x_r + 2V(j)T_r)v\ dx^r.$$

If δ is an infinitesimal W-rotation, we have

$$\delta u(j) = W_r(jk)u(k)dx^r = \bar{v}'W_r(jk)V(k)v\ dx^r.$$

Hence, the relations

$$2V(j)T_r = W_r(jk)V(k) - \partial V(j)/\partial x_r,$$

$$2V(j)V(j)T_r = W_r(jk)V(j)V(k) - V(j)\partial V(j)/\partial x_r$$

are valid. Now we have to calculate the invariant $V(j)V(j)$. We obtain the following result:

$$V(j)V(j) = a_p(j)a_q(j)v^pv^q = g_{pq}v^pv^q = \bar{H}H.$$

Therefore, we may write

$$T_r = H^{-1}\bar{H}^{-1}(W_r(jk)V(j)V(k) - V(j)\partial V(j)/\partial x_r)/2.$$

Following H. Weyl, we define

$$Dv = dv + \delta Tv,$$

$$D_r v = (\partial/\partial x_r + T_r)v.$$

This definition of covariant spinor differentiation is much more general than that by H. Weyl, where the matrices $V(j)$ are constant. Our definition seems to be the most general one that is possible at all.

11. Schwarzschild space

In this chapter we shall illustrate some of our theoretical conceptions by the example of the Schwarzschild space. The quadratic differential form

$$(ds)^2 =$$
$$= -(1 - 2m/r)(dt)^2 + (1 - 2m/r)^{-1}(dr)^2 +$$
$$+ r^2(du)^2 + r^2\sin^2 u\,(dv)^2$$

is the Schwarzschild metrical fundamental form. On introducing the coordinates $x_1 = t$, $x_2 = r$, $x_3 = u$, $x_4 = v$, we obtain the matrix

$$G = \begin{pmatrix} -(1-2m/r) & 0 & 0 & 0 \\ 0 & (1-2m/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2 u \end{pmatrix}$$

as the metrical fundamental matrix of Schwarzschild space. The matrix

$$A = \begin{pmatrix} i(1-2m/r)^{1/2} & 0 & 0 & 0 \\ 0 & (1-2m/r)^{-1/2} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin u \end{pmatrix}$$

is uniquely determined by the metrical decomposition

$$G = AA'.$$

The matrices

$$U = i2^{-1/2} \begin{pmatrix} i(1-2m/r)^{1/2} & 0 & 0 & r \sin u \\ 0 & (1-2m/r)^{-1/2} & ir & 0 \\ 0 & (1-2m/r)^{-1/2} & -ir & 0 \\ i(1-2m/r)^{1/2} & 0 & 0 & -r \sin u \end{pmatrix}$$

and

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

lead to a metrical decomposition

$$G = U'(\bar{H} \otimes H)U.$$

From

$$V = -i2^{-1/2} \begin{pmatrix} i(1-2m/r)^{-1/2} & 0 & 0 & i(1-2m/r)^{-1/2} \\ 0 & (1-2m/r)^{1/2} & (-2m/r)^{1/2} & 0 \\ 0 & -i/r & i/r & 0 \\ 1/r \sin u & 0 & 0 & -1/r \sin u \end{pmatrix}$$

it follows that

$$V^1 = - 2^{-1/2} i(1 - 2m/r)^{-1/2} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} ,$$

$$V^2 = - 2^{-1/2} i(1 - 2m/r)^{1/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$V^3 = - 2^{-1/2} i/r \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ,$$

$$V^4 = - 2^{-1/2} i/r \sin u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Conversion of indices yields the matrices

$$V(1) = -i2^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$V(2) = -i2^{-1/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$V(3) = -i2^{-1/2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ,$$

$$V(4) = -i2^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

This shows that in the case of Schwarzschild space our theory of covariant spinor differentiation does not differ from Weyl's theory. The same holds for all diagonal metrical fundamental forms.

13. The decomposition of the Gordon operator

1. Introduction:

Dirac's celebrated method of deriving the euqation of the electron was based on the decomposition of the Gordon operator in the form: $(\Box - m^2)\Psi = (\sum_i \gamma_i \partial_i + m)(\sum_i \gamma_i \partial_i - m)\Psi = 0$ (1),

which led him automatically to both the known Dirac matrices with their commutation relations, and to the Dirac equation. By a decomposition of this kind it is obviously possible to reduce a second order differential equation (Klein-Gordon Eq.) to a differential equation of the first order (Dirac Eq.). This was so far the decisive point since positive definiteness of the probability density and relativistic covariance postulated a differential equation of the first order.

To find the equation of the electron (and also of other elementary particles) in the general theory of relativity it appears obvious to proceed analogously, i.e. to decompose the Gordon operator and reduce it to a form which is analogous to (1). In the following, the possibilities and conditions of a decomposition of that kind are studied more thoroughly and are then discussed. *)

2. Decomposition of the Gordon operator:

In the general theory of relativity equation (1) has the following form:

$$(2) \quad (\Box - m^2)\Psi = (\sum_{i,k} g^{ik} D_i D_k - m^2)\Psi = (\sum_i V^i D_i + m)(\sum_k V^k D_k - m)\Psi = 0$$

---

*) At this place we shall notice that our derivation of Dirac's equation in the First Final Technical Report is wrong because it was based on an uncritical concept of covariant spinor differentiation.

where D is a differential operator acting on the wave function Ψ, (it may e.g. be the covariant differentiation, but it may also be much more general. Any assumption in point is withheld here intentionally).

If we now form the product $(\sum_i V^i D_i + m)(\sum_k V^k D_k - m)$

we arrive at:

$$(3) \quad \frac{1}{2} \sum_{i,k} (V^i D_i V^k D_k + V^k D_k V^i D_i) - m^2$$

If, then, we postulate that $D_i V^k = V^k D_i$ for all i and k, it is found that (3) passes over into

$$(3') \quad \frac{1}{2} \sum_{i,k} (V^i V^k D_i D_k + V^k V^i D_k D_i)$$

For the purpose of making a comparison with the left side of (2) we write:

$$(4) \quad \Box = \sum g^{ik} D_i D_k = \frac{1}{2} \sum (g^{ik} D_i D_k + g^{ki} D_k D_i)$$

The comparison of (3') with (4) gives:

$$(5) \quad V^i V^k = g^{ik}$$

It must, however, be mentioned that this derivation includes the assumption that $D_i D_k \neq D_k D_i$, which usually applies in the general theory of relativity. If $D_i D_k = D_k D_i$ (as e.g. in the Lorentz metrics), (5) is replaced by the condition $(V^i V^k + V^k V^i) = 2g^{ik}$, as can readily be seen from (3') and (4),

Hence, for a decomposition in the form of (2), the following conditions must be satisfied:

$$\left.\begin{array}{lll} \text{(I)} & D_i V^k = V^k D_i & \\ & & \text{for all } i \text{ and } k \\ \text{(II)} & V^i V^k = g^{ik} & \end{array}\right.$$

(II) can also be written in the form $\underline{V_i V_k = g_{ik}}$, since

$\underline{V^iV^k} = g^{il}V_l g^{ku}V_u = g^{il}g^{ku}V_l V_u = g^{il}g^{ku}g_{lu} = g^{il}g^k_l = \underline{g^{ik}}$, V^i thus being equal to $g^{il}V_l$.

3. Discussion:

a) It is easily seen that $V^iV^k + V^kV^i = 2g^{ik}$ follows from (II), since $g^{ik} = g^{ki} = V^{ki}$, and hence $V^iV^k + V^kV^i = g^{ik} + g^{ki} = 2g^{ik}$. It must, however, be noted that this conclusion cannot be reversed. The condition (II) thus implies a restriction for the V^i which is essentially greater than that of the usual condition $V^iV^k + V^kV^i = 2g^{ik}$.

b) From (II) it can further be derived immediately that

$$V^iV^k = V^kV^i, \text{ hence } \left[V^i, V^k\right] = 0$$

c) From (I) and (II) it follows moreover that $g^{ik}D_i = D_i g^{ik}$ or that $\underline{D_i g^{ik} = 0}$, since $V^iV^kD_i = g^{ik}D_i = D_iV^iV^k = D_ig^{ik}$. (6)

That means that if D_i is assumed to be the covariant differential operator for a tensor of the second rank, (6) represents nothing else but the Ricci theorem of the general theory of relativity. The conditions (I) and (II) thus appear to be consistent with the general theory of relativity.

d) It is above all the condition (II) that seems to postulate rather too much since it fails to be satisfied already in the case in which the metric is diagonal. For if the metric is assumed to be composed of the diagonal elements a_1, a_2, a_3, a_4, the following relation must necessarily be valid according to (II):

$$V_i^2 = g_{ii} = a_i \quad \text{and hence}$$

$$(\det V_i)^2 = (\det g_{ii}) \text{ , i.e. } \det V_i = \pm\sqrt{(\det g_{ii})} \neq 0$$

for all i (x)

But on the other hand also $V_i V_k = g_{ik} = 0$ must be satisfied for $i \neq k$. From this it can be concluded that either det V_i

or det V_k must be zero, which is inconsistent with (x).

4. A method for decomposing the Gordon operator so that $V^i V^k + V^k V^i = 2g^{ik}$:

In the following an attempt is made to decompose the Gordon operator, retaining the commutation relation: $V^i V^k + V^k V^i = 2g^{ik}$. We are prompted by two reasons to proceed in this way: On the one hand it appears extremely difficult, and most probably even impossible, to satisfy the conditions (I) and (II), as is apparant from 3d), on the other hand, the natural extension of the ordinary Dirac commutation relations

$V_i V_k + V_k V_i = 2\delta_{ik}$ to the general theory of relativity is given

by $\delta_{ik} \rightarrow g_{ik}$, so that

$$V_i V_k + V_k V_i = 2g_{ik} \quad \text{or}$$

$V^iV^k + V^kV^i = 2g^{ik}$ are obtained directly ($V^i = g^{il} V_l$).

Here we start again from (2) and (3): Hence

$$(7) \quad (\sum_i V^i D_i + m)(\sum_k V^k D_k - m) = \frac{1}{2} \sum_{i,k} (V^i D_i V^k D_k + V^k D_k V^i D_i) - m^2$$

We must now again postulate that $D_i V^k = V^k D_i$, writing at the same time additionally $D_k D_i = D_i D_k + A_{ki}$ (the A_{ki} will be determined more accurately later). (See appendix). Then (7) passes over into

$$(8) \quad (\sum_i V^i D_i + m)(\sum_k V^k D_k - m) = \frac{1}{2} \sum_{i,k} (V^i V^k + V^k V^i) D_i D_k$$

$$+ \frac{1}{2} \sum_{i,k} V^k V^i A_{ki} - m^2$$

We now assume that $V^i V^k + V^k V^i = 2g^{ik}$ and that the Klein-Gordon Eq. is valid: $(\Box - m^2)\Psi = (\sum g^{ik} D_i D_k - m^2)\Psi = 0$

Hence (8) reads to

$$(9) \qquad \sum_{i,k} V^k V^i A_{ki} = 0$$

which means that if condition (9) is satisfied, it will be possible to decompose the Gordon operator and reduce it to the form of (2) and hence to a Dirac equation

$(\sum_i V^i D_i - m)\Psi = 0$ using $V^i V^k + V^k V^i = 2g^{ik}$ and $\sum_{i,k} V^k V^i A_{ki} = 0$

5. Conclusions:

It is possible to decompose the Gordon operator in the form

$$(\Box - m^2)\chi = (\sum_i V^i D_i - m)\,(\sum_k V^k D_k + m):$$

1) If $\begin{cases} D_i V^k = V^k D_i \\ V^i V^k = g^{ik} \end{cases}$

2) Or if $\left.\begin{array}{l} D_i V^k = V^k D_i \\ V^i V^k + V^k V^i = 2g^{ik} \\ \text{and} \quad \sum_{i,k} V^k V^i A_{ki} = 0 \end{array}\right\}$, A_{ki} being defined by $D_k D_i =$

$= D_i D_k + A_{ki}$.

Appendix.

Determination of the A_{ki}:

From the equation defining the A_{ki}:

(10) $\qquad D_k D_i = D_i D_k + A_{ki}$

it can be derived immediately that the A_{ki} must be anti-symmetrical.

For if in (10) i and k are commutated one obtains

$$D_i D_k = D_k D_i + A_{ik} \quad \text{or} \quad D_k D_i = D_i D_k - A_{ik}. \tag{11}$$

Hence the comparison with (10) entails the antisymmetry

$A_{ki} = - A_{ik}$. It must be pointed out that, in the case in which the D_i indicate a covariant differentiation and $D_k D_i$ acts on a vector, A_{ik} is the Riemann-Christoffel curvature tensor.

If we assume for D_i quite generally the form:

(11) $D_i = \partial_i + C_i$ we obtain for $D_k D_i$:

$$D_k D_i = D_i D_k - \partial_i C_k - C_i \partial_k - C_i C_k + \partial_k C_i + C_k \partial_i + C_k C_i =$$

$$= D_i D_k + (C_k \partial_i - C_i \partial_k) + (\partial_k C_i - \partial_i C_k) + (C_k C_i - C_i C_k)$$

i.e. for the A_{ki}:

(12) $A_{ki} = (C_k \partial_i - C_i \partial_k) + (\partial_k C_i - \partial_i C_k) + (C_k C_i - C_i C_k)$,

which also shows the antisymmetry at first sight. (12) shows that the A_{ki} will be determined as soon as the differential operator D_i is known.

14. Development of the spinor theory in the Riemannian geometry during the last years

A. Introduction:

In this chapter a survey will be given on the efforts made and the developments in the spinor theory in the general theory of relativity in the course of the recent years.

Since about 1930 when Fock, Iwanenko, Weyl, Schrödinger, Infeld and Van der Waerden [4] attempted for the first time a generalization of spinors and of the Dirac theory of the electron to the Riemannian geometry and obtained fundamental results, no essentially new results had been obtained in this field for a long time.

Only less than ten years ago interest in this field grew again in order to study the relationship between the field theory and the elementary particle theory and the quantum electrodynamics on the one hand and the general theory of relativity on the other.

What are the main problems that remained unsolved in 1930 and which, therefore required a more detailed investigation in the last years and even now are not completely solved ?

(1) In most of the studies with the what is called "four-leg" formalism a special system of co-ordinates was used. Only Schrödinger [3] based the calculations on fully general, curvilinear co-ordinates. He did not succeed, however, in establishing a completely consistent spinor theory. (See the following points.)

(2) There was no general analytical expression for the spinor affinity in <u>arbitrary</u> curvilinear co-ordinates. Hence the analysis was widely uncertain.

(3) The hermiticity properties of the γ-matrices required by most of the authors were admissible only for special systems of co-ordinates but not for general ones. As a result, difficulties arose later in the formation of covariant expressions.

(4) The relationship between spinor theory and bispinor theory including the relationship between the γ-matrices and the

σ-matrices was not known at all.

(5) The behavior of spinor equations towards P-, C-, T- transformations which play an important part in the theory of elementary particles has not yet been discussed sufficiently. Above all the problem of the violation of parity in the general theory of relativity has hitherto not been explained.

All these shortcomings might perhaps be explained best by the lack of an axiomatic theory of spinors in the Riemannian space. Above all, E. Schmutzer [5,6,7,8] dealt with this axiomatic theory and made studies on spinor algebra and spinor analysis in this sense. He also succeeded in solving a large number of the above problems. The following considerations are based on his studies.

Also the papers of P. Bergmann [9], Green [10], [11], Fletcher [12], Nakamura and Toyoda [13], Stephenson [14], Higgs [15] and others are worth being mentioned. They also contributed to the solution of the problems mentioned above.

The following pages give a survey on the results of the most important studies.

B. Spinor algebra in the Riemannian space:

1. Fundamental conceptions:

Denotations: Greek indices refer to the real tensor space which shall have the signature (+,+,+,-).

Latin capital indices refer to the twodimensional spinor space.

The following formulas are defined in simple extension to the corresponding formulas in the Minkowski space.

It has been found that spinor algebra as well as spinor analysis can be constructed fully consistently.

a) The metric in the spinor space shall be given by h_{AB}, so that for a spinor u:

$$\begin{cases} u_A = h_{BA} \; u^B \\ u^A = h^{AB} \; u_B \end{cases}$$

where $h_{AB}\, h^{CB} = h_A{}^C = \delta_A{}^C$; $h_{AB} = -h_{BA}$

The behavior for automorphic transformations of a spinor is described by

$$u^{A'} = A^{A'}{}_A\, u^A$$

$$u_{A'} = A_{A'}{}^A\, u_A$$

with $A^{A'}{}_B\; A_{A'}{}^C = h_B{}^C$ and $A^B{}_{A'}\; A^{A'}{}_A = h_A{}^B$

from which we also obtain

$$A^{A'}{}_B = A_B{}^{A'} \equiv A^{A'}_B$$

$$A_{A'}{}^B = A^B{}_{A'} \equiv A^B_{A'}$$

b) The relationship between vectors (generally tensors) and spinors, in analogous extension to the Lorentz case, is given by:

$$a^\mu = \frac{1}{2}\, \sigma^{\mu\dot{A}B}\, u_{\dot{A}B} \quad \text{and the inversion}$$

$$u_{\dot{A}B} = -a^\mu\, \sigma_{\mu\dot{A}B}$$

$\sigma^{\mu\dot{A}B}$ is called the metric spin tensor

2. <u>Fundamental axioms for the construction of the spinor algebra:</u>

The whole algebra is dominated by 2 axioms:

<u>First Axiom:</u> Hermiticity of the metric spin tensor:

$$\underline{\sigma^{\mu\dot{A}B} = \sigma^{\mu\dot{B}A}}$$

Second Axiom:

$$\sigma_{\mu\ A}^{\ \ \dot{B}}\ \sigma_{\nu\ \dot{B}C} = g_{\mu\nu}\, h_{AC} \ \pm\ \frac{i}{2}\ \mathcal{E}_{\mu\nu\varrho\tau}\ \sigma^{\varrho\dot{B}}{}_{A}\ \sigma^{\tau}{}_{\dot{B}C}$$

($\mathcal{E}_{\mu\nu\varrho\tau}$ . . . Levi - Cività's pseudo-tensor)

The second axiom is substantiated by splitting the spinor product $\sigma_{\mu\ A}^{\ \ \dot{B}}\ \sigma_{\nu\dot{B}C}$ into one part symmetrical in $\mu\nu$ and into another antisymmetrical in $\mu\nu$, in the following form:

$$\sigma_{\mu\ A}^{\ \ \dot{B}}\ \sigma_{\nu\ \dot{B}C} = A g_{\mu\nu}\ h_{AC} + B\mathcal{E}_{\mu\nu\varrho\tau}\ \sigma^{\varrho\dot{B}}{}_{A}\ \sigma^{\tau}{}_{\dot{B}C}$$

For simplicity A is normalized to 1. B is then obtained from the inner consistency of the axiom (left-hand side of the equation is substituted in the right-hand side).
The following important and interesting relationships follow from the two axioms:

$$\sigma_{\mu\ A}^{\ \ \dot{B}}\ \sigma_{\nu\dot{B}A} = -\ 2g_{\mu\nu}$$

$$\sigma_{\mu\ A}^{\ \ \dot{B}}\ \sigma^{\mu}{}_{\dot{B}C} = 4h_{AC}$$

$$\sigma^{\nu}{}_{\dot{A}B}\ \sigma_{\nu\dot{C}D} = -2h_{\dot{A}\dot{C}}\ h_{DB}$$

and $$\sigma_{\nu\dot{A}B}\ \sigma_{\mu\dot{C}D}\ g^{\nu\mu} = -2\ h_{\dot{A}\dot{C}}\ h_{BD}$$

The last formulas are derived mainly by interchanging in the second axiom μ and ν, then once adding this expression to, once subtracting it from the second axiom and then multiplying it with adequate factors.

C. Spinor analysis in the Riemannian space:
A covariant derivative of the spinors shall be defined in the

Riemannian geometry. For only spinor algebra and spinor analysis make it possible to establish spinor equations.

Definition of the covariant derivative:

$$u_{A;\nu} = u_{A,\nu} - \Gamma^{B}_{A\nu}\, u_B$$

$$u^{A}{}_{;\nu} = u^{A}{}_{,\nu} + \Gamma^{A}_{B\nu}\, u^B$$

$\Gamma^{B}_{A\nu}$ . . . spinor affinity

The purpose is to find an expression for the spinor affinity. The following axiom (Third Axiom) shall be added to the above definition with which spinor analysis will then be set up:

$$\underline{\sigma^{\nu\dot{A}B}{}_{;\mu} = 0}$$

Hence it determines the covariant derivative of the metric spin tensor.

The definition of the covariant derivative gives the following relations for the covariant derivative of the metric in the spinor space:

$$h_{A}{}^{B}{}_{;\nu} = 0$$

$$h_{AB;\nu} = h^{CD}{}_{;\nu}\, h_{AC}\, h_{DB}$$

$$h^{AB}{}_{;\nu} = -\, h_{CD;\nu}\, h^{DB}\, h^{CA}$$

With the abbreviation

$$h_{\dot{A}\dot{B};\nu}\, h^{\dot{A}\dot{B}} = 2i\, \overset{*}{\Phi}_{\nu},$$

the following relations are obtained with the aid of the second axiom:

$$h_{AB;\nu} = - i\, \Phi_{\nu}\, h_{AB}$$

$$h^{AB}{}_{;\nu} = i\, \Phi_{\nu}\, h^{AB}$$

Hence these equations determine the covariant derivative of the metric spinor.

Since it would be too long to give all formulas and derivatives which eventually lead to an explicit expression for the affinity, we shall briefly outline the method:

$\Gamma^{B}_{A\nu}$ is split in the following form

$$\Gamma^{B}_{A\nu} = \begin{bmatrix} B \\ A\nu \end{bmatrix} + \frac{i}{2}\, h_{A}{}^{B}\, \Phi_{\nu}$$

The significance of this splitting will become manifest only in the following considerations. The following conditions for the $[\cdot/\cdot]$ follow directly from the formulas derived above:

$$\left.\begin{aligned} h_{AB,\nu} &= h_{CB} \begin{bmatrix} C \\ A\nu \end{bmatrix} - h_{CA} \begin{bmatrix} C \\ B\nu \end{bmatrix} \\ h^{AB}{}_{,\nu} &= - h^{CB} \begin{bmatrix} A \\ C\nu \end{bmatrix} - h^{AC} \begin{bmatrix} B \\ C\nu \end{bmatrix} \end{aligned}\right\}$$

besides

$$\begin{bmatrix} A \\ A\nu \end{bmatrix} + \begin{bmatrix} \dot{A} \\ \dot{A}\nu \end{bmatrix} = 2\Gamma_{,\nu}\,, \quad \text{where} \quad \Gamma = \text{en}\sqrt{h}; \quad h = h_{\dot{1}\dot{2}}\, h_{12}$$

(the latter formula is the relation following from the definition of the covariant derivative:

$$\Gamma^{A}_{A\nu} + \Gamma^{\dot{A}}_{\dot{A}\nu} = (\ln h)_{,\nu} - \frac{1}{h}\, h_{;\nu}$$

$\begin{bmatrix} A \\ A\nu \end{bmatrix}$ shall be split into real and imaginary part:

$\begin{bmatrix} A \\ A\nu \end{bmatrix} = \Gamma_{,\nu} + i\,\Pi_\nu$ (for significance of Π_ν see below)

The following explicit expression can be obtained for the spinor affinity: (for more detailed derivation see the original paper [5]):

$$\Gamma^{B}_{A\nu} = \begin{Bmatrix} B \\ A\nu \end{Bmatrix} + \frac{i}{2} h_A{}^B \left(\Phi_\nu + \Pi_\nu + i\,\Gamma_{,\nu} \right)$$

where

$$\begin{Bmatrix} B \\ A\nu \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} \sigma_{\lambda\dot{C}A}\, \sigma^{\mu\dot{C}B} - \frac{1}{4} \sigma^{\mu\dot{C}B}\, \sigma_{\mu\dot{C}A,\nu}$$

<u>Significance of Π_ν :</u>

According to E. Schmutzer (in contrast to Van der Waerden and Infeld) Π_ν can be related to the four-potential A_ν:

$$A_\nu = \frac{\hbar c}{2e} \Pi_\nu$$

since in a phase transformation (rotation in the spinor space) $A_A^{B'} = \delta_A{}^B e^{i\varphi/2}$, Π_ν transforms like the electromagnetic potential:

$$A_\nu \rightarrow A_\nu - \frac{\hbar c}{2e} \varphi_{,\nu}$$

$$\Pi_\nu \rightarrow \Pi_\nu - \varphi_{,\nu}$$

In this respect the spinor affinity is thus determined. Of course we can write $\Pi_\nu = 0$ (space without electromagnetic field).

D. Dirac equation:

To develop a Dirac equation in the Riemannian space two independent ways are possible, in analogy to the Lorentz space, e.g. that of Infeld and Van der Waerden [16] which is based on the spinor theory, and that of the bispinor theory which Schrödinger [3] used for the first time. However, he did not succeed in developing it fully consistently.
E. Schmutzer [5,7,8] coordinated these two forms.
(1) For this purpose it is important to construct a theory of the Dirac γ-operators:
Unlike most of the authors which based the theory on the Dirac γ-matrices, hence regarded it as primary and the metric as secondary due to the relation
$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$, we shall integrate the γ-operators

into E. Schmutzer's axiomatic theory [7]. The commutation relation shall, however, be valid:
Since, like in the Lorentz metric, the $\sigma^{\mu A\dot{B}}$ are very probably (as will be proved later on, related to the Pauli matrices, the following extremely general statement is made upon which also the second axiom is based:

$$\gamma_\mu \gamma_\nu = g_{\mu\nu} + \varepsilon_{\mu\nu}{}^{\alpha\beta} \gamma_\alpha \gamma_\beta C$$

where C (an operator) is still unknown, but is determined to be $-4C^3 = C$, if $\gamma_\alpha \gamma_\beta$ is again expressed by the same formula. γ_5 can be defined in analogy to the Lorentz case:

$$\begin{cases} \gamma_5 = \dfrac{1}{4!i} \quad \varepsilon^{\nu\mu\alpha\beta} \gamma_\nu \gamma_\mu \gamma_\alpha \gamma_\beta \\ \gamma_5{}^2 = 1 \end{cases}$$

Eventually, $C = \frac{1}{2} i\gamma_5$ is obtained so that the above statement

can be written as follows and at the same time regarded as the fourth axiom:

$$\underline{\gamma_\mu \gamma_\nu = g_{\mu\nu} + \frac{1}{2}\, i\, \varepsilon_{\mu\nu}{}^{\alpha\beta}\, \gamma_\alpha \gamma_\beta \gamma_5}$$

Besides, also $\gamma^\mu = g^{\mu\nu}\gamma_\nu$ shall be defined.
It is evident at first sight that the generally known commutation relation $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ is a direct consequence of this axiom.
In the theory of γ-operators its <u>hermiticity conditions</u> are of great importance.
It has been found that the simple extension of hermiticity in the Lorentz space to the Riemannian space

$$\begin{aligned} \gamma_\nu{}^+ &= \gamma_\nu \\ \gamma_4{}^+ &= \alpha\gamma_4 \quad (\alpha \text{ real}) \end{aligned} \qquad (*)$$

is <u>impossible</u> in the framework of this <u>axiomatic theory</u> but that either only

$$\gamma^{\nu+} = \gamma^\nu \quad \text{and} \quad \gamma_4{}^+ = -\gamma_4$$

$$\text{or} \quad \gamma_\nu{}^+ = \gamma_\nu \quad \text{and} \quad \gamma^{4+} = -\gamma^4$$

is compatible with the previously derived conditions and in this metric. In both cases, however, it follows that $\gamma_5{}^+ = \gamma_5$.

(2) In order to establish the relationship between spinor geometry and bispinor geometry the problem of the splitting of γ-operators into the Pauli σ-operators has still to be solved.
If the most general formulation

$$\gamma^\mu = i \begin{pmatrix} \alpha^\mu, & -\sigma^\mu \\ \varrho^\mu, & \beta^\mu \end{pmatrix} ,$$

which may bring about such a splitting, enters the above formulas for the γ and $\sigma^{\mu A\dot{B}}$ etc. then we obtain $\alpha^\mu = \beta^\mu = 0$ in any case.

Thus, it follows generally that

$$\gamma^{\mu} = i \begin{pmatrix} 0, & -\sigma^{\mu} \\ \varrho^{\mu}, & 0 \end{pmatrix}$$ with the relations

$$\left.\begin{aligned} \sigma^{\mu}\varrho^{\nu} + \sigma^{\nu}\varrho^{\mu} &= 2g^{\mu\nu} \\ \varrho^{\mu}\sigma^{\nu} + \varrho^{\nu}\sigma^{\mu} &= 2g^{\mu\nu} \end{aligned}\right\}$$

where $\sigma^{\mu} = \sigma^{\mu A\dot{B}}$

and $\varrho^{\mu} = -\sigma^{\mu}{}_{\dot{A}B}$

For γ_5 we obtain the splitting $\gamma_5 = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(It should be **reminded** that 1 is the binary unit matrix)
If the above hermiticity condition (x) is imposed on the γ-matrices, then the following simple formulas are obtained:

$$\sigma^{i} = \varrho^{i}, \quad \sigma_4 = -\varrho_4; \quad h = 1, \quad \Gamma_{,\mu} = 0$$

i.e., furthermore

$$\left.\begin{aligned} \sigma^{i}\sigma^{j} + \sigma^{j}\sigma^{i} &= 2g^{ij} \\ \sigma^{i}\sigma_4 &= \sigma_4\sigma^{i} \\ (\sigma_4)^2 &= -g_{44} \end{aligned}\right\}$$

(3) Construction of the Dirac equation:

(a) Spinor theory:

According to Van der Waerden [16] the Dirac equation in Riemannian geometry has the following form:

$$\begin{cases} \sigma^{\mu\dot{A}B}\,\Psi_{B;\mu} - i\,\varkappa\,\chi^{\dot{A}} = 0 & \varkappa = \frac{mc}{\hbar} \\ \sigma^{\mu}{}_{\dot{B}A}\,\chi^{\dot{B}}{}_{;\mu} - i\,\varkappa\,\Psi_{A} = 0 & \end{cases}$$

where $\Psi_{B;\mu}$ was determined in chapter C:

$$\left.\begin{aligned} \Psi_{B;\mu} &= \Psi_{B,\mu} - \Gamma^{A}_{B\mu}\,\Psi_{A} \\ \chi^{B}{}_{;\mu} &= \chi^{B}{}_{,\mu} + \Gamma^{B}_{A\mu}\,\chi^{A} \\ \text{where} \quad \Gamma^{B}_{A\mu} &= \left\{ {B \atop A\mu} \right\} + \frac{1}{2}\,i\,h_{A}{}^{B}\,(\Phi_{\mu} + \Pi_{\mu} + i\Gamma_{,\mu}) \end{aligned}\right\}$$

(For more detailed significance of the individual quantities see chapter C). $\Gamma_{,\mu} = 0$, if hermiticity is imposed on the γ-matrices in the above sense.

$$\Pi_{\mu} = \frac{2\epsilon}{\hbar c}\,A_{\mu} \qquad A_{\mu} \text{ - four potential}$$

The physical significance of Φ_{μ} has hitherto not been fully explained.

(b) Bispinor theory:

In this theory the Dirac equation can be written in the following form [8]:

$$\gamma^{\mu}\,\Psi_{;\mu} + \varkappa\,\Psi = 0$$

where

$$\gamma^{\mu} = i\begin{pmatrix} 0, & -\sigma^{\mu} \\ \varrho^{\mu}, & 0 \end{pmatrix} \quad ; \quad \begin{cases} \sigma^{\mu} = \sigma^{\mu A\dot{B}} \\ \varrho^{\mu} = -\sigma^{\mu}{}_{\dot{A}B} \end{cases}$$

and $\quad \Psi_{;\mu} = \Psi_{,\mu} + \Gamma_{\mu}\Psi \qquad \Psi$ - bispinor

Both forms become identical if

$$\Psi = \begin{pmatrix} \chi^{A} \\ \psi_{\dot{A}} \end{pmatrix} \qquad \Psi^{+} = (\chi^{\dot{A}}, \Psi_{A})$$

and
$$\Gamma_{\mu} = \begin{pmatrix} \Gamma^{A}_{B\mu} \, , & 0 \\ 0 \, , & -\Gamma^{\dot{B}}_{\dot{A}\mu} \end{pmatrix}$$

$$\Gamma_{\mu}^{+} = \begin{pmatrix} \Gamma^{\dot{B}}_{\dot{A}\mu} \, , & 0 \\ 0 \, , & -\Gamma^{A}_{B\mu} \end{pmatrix}$$

Adjoint Dirac equation:

In analogy to the Lorentz case it is

$$\bar{\Psi}_{;\mu}\,\gamma^{\mu} - \varkappa\,\bar{\Psi} = 0$$

$$\bar{\Psi}_{;\nu} = \bar{\Psi}_{,\nu} - \bar{\Psi}\,\Gamma_{\nu}$$

Continuity equation:

$$(\bar{\Psi}\,\gamma^{\mu}\,\Psi)_{;\mu} = 0$$

where $\qquad \bar{\Psi} = \Psi^{+}\beta$

β is determined from the adjoint equation, the continuity equation and the covariant differentiation of the bispinor with the exception of a constant factor, which, in this case, was put equal to 1:

$$\beta = \begin{pmatrix} 0 \, , & h_{\dot{A}}{}^{\dot{B}} \\ -h^{A}{}_{B} \, , & 0 \end{pmatrix} ; \qquad \beta^{2} = 1, \; \beta^{+} = \beta$$

E. Formation of covariants:

The problem of the formation of covariant expressions is of special importance in formulations in the field theory and the theory of elementary particles. For this reason the formation of covariant expressions within the framework of this theory shall be dealt with briefly.
In the sense of the spinor theory developed in B. and C. the transformation behavior with respect to spinor and tensor space can be read easily from the above index form and, therefore, needs no further discussion.
This is also possible easily within the framework of the bispinor theory if the relations between spinor and bispinor theory given in the last chapter are used. With the aid of these expressions the bispinor theory can be reduced to the spinor theory.
In this way the transformation behavior of the following quantities can be understood easily:

$\bar{\Psi}\,\Psi$ . . . . hermitian, <u>scalar</u>

$(i\,\bar{\Psi}\,\gamma^{\mu}\,\Psi)$ . . . . <u>vector</u>, hermitian

$(i\,\bar{\Psi}\,\gamma_5\,\Psi)$ . . . . <u>pseudoscalar</u>, hermitian

etc. i.e. expressions, fully analogous to the Lorentz case. It is pointed out that for this purpose no hermiticity conditions are imposed on γ^{μ}.

F. Generalized Dirac equation according to E. Cartan

It is of some interest to compare E. Cartan's method of "repere mobile" with H. Weyl's treatment as represented in the preceding two chapters. Cartan writes the Dirac equation in the form

$$((h/i)D - m_0 K)u = 0,$$

where

$$K = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

D is the operator of covariant differentiation. Its effect onto a

vector x is the following:

$$Dx = dx + \delta x, \qquad (11)$$

in which d is the relative differentiation, and δ is a certain infinitesimal rotation. The vector x corresponds to a spinor quantity X of the second rank. The co-ordination $x \to X$ is such that

$$(xy) = (XY + YX)/2, \qquad (12)$$

where (xy) is the inner product of the vectors x and y. The relation (12) is the fundamental formula of Cartan's spinor calculus. The effect of the operator D onto a spinor X is

$$DX = dX + \delta X \qquad (13)$$

in which

$$\delta X = (X\delta\Omega - \delta\Omega X)/2 \qquad (14)$$

and

$$\delta\Omega^* + \delta\Omega = 0. \qquad (15)$$

These relations are in full agreement with the case of an euclidean metrical ground form. The transition from a spinor X of the second rank to a spinor u of the first rank is symbolically done as follows:

$$X = uu^*. \qquad (16)$$

The infinitesimal rotation corresponding to the infinitesimal rotation δX is

$$\delta u = -\delta\Omega u/2. \qquad (17)$$

Proof:

$$\delta X = \delta uu^* + u\delta u^* = -\delta\Omega uu^*/2 - uu^*\delta\Omega^*/2 = (X\delta\Omega - \delta\Omega X)/2.$$

Therefore we obtain

$$Du = du - \delta\Omega u/2.$$

The infinitesimal matrix $\delta\Omega$ depends on the metrical ground form. Its calculation is formally similar to that of Weyl's dE. Its meaning is, however, somewhat different from that of dE.

G. Remarks to the preceding report on recent literature

The investigations referred to in the preceding chapter have some features in common with our own investigations. The most remarkable of them is that covariant spinor differentiation is understood in the sense of the tensor space and not in that of the spinor space. This has been done for the first time by H. Weyl after E. Cartan had shown that the other way was impossible. There is only one possibility of generalizing Weyl's concept of spinor differentiation: variable spinor metrical fundamental form H. This possibility has been used by the authors mentioned as well as by ourselves. Our fundamental assumptions, however, are much more general than those of other authors. Therefore, our results do not go into such detail and are not yet capable of special physical interpretation. Some of the axioms introduced by other authors seem, however to be artificial and not suggested by geometrical facts.

15 Mathematical appendix: A quadratic calculus

This chapter contains a calculus similar to the Kronecker one. We have developed it in order to investigate spinor algebras in a formally simple manner. Let u and v be two binary vectors the elements of which are operators:

$$u = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \qquad v = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} .$$

Let A be a binary matrix with real or complex elements,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ,$$

which interrelates the vectors u and v:

$$v = Au.$$

By means of the matrix

$$R = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

we define the squares

$$U^{(2)} = R(u \otimes u) = \begin{pmatrix} 2U_1^2 \\ U_1U_2 + U_2U_1 \\ 2U_2^2 \end{pmatrix}$$

and

$$A^{(2)} = \begin{pmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{pmatrix} .$$

Now we have

$$R(A\otimes A) = \begin{pmatrix} 2a_{11}^2 & 2a_{11}a_{12} & 2a_{12}a_{11} & 2a_{12}^2 \\ 2a_{11}a_{21} & a_{11}a_{22}+a_{21}a_{12} & a_{11}a_{22}+a_{12}a_{21} & 2a_{12}a_{22} \\ 2a_{21}^2 & 2a_{21}a_{22} & 2a_{22}a_{21} & 2a_{22}^2 \end{pmatrix} = A^{(2)}R.$$

From this basic relation we derive the following:

$v^{(2)} = R(v\otimes v) = R(A\otimes A)(u\otimes u) = A^{(2)}R(u\otimes u) = A^{(2)}u^{(2)}.$

As an example we consider a two-component model of our derivation of generalized Pauli and Dirac matrices. We assume that the operators U_1, U_2 fulfil the commutation relations

$$U_iU_k + U_kU_i = 2\delta_{ik},$$

which we may write in the form

$$u^{(2)} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then we have

$$v^{(2)} = 2\begin{pmatrix} a_{11}^2 + a_{12}^2 \\ a_{11}a_{21} + a_{12}a_{22} \\ a_{21}^2 + a_{22}^2 \end{pmatrix}.$$

The requirement that

$$V_iV_k + V_kV_i = 2g_{ik},$$

or

$$v^{(2)} = 2\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix},$$

leads to the conditions

$$a_{11}^2 + a_{12}^2 = g_{11},$$
$$a_{11}a_{21} + a_{12}a_{22} = g_{12},$$
$$a_{21}^2 + a_{22}^2 = g_{22},$$

which may be written in the form

$$AA' = G.$$

This result is well-known to us.

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17. Administrative Data.

The research work described in this document has been made by Prof.Dr. Ferdinand Cap (approximately 35o hrs) and by Dr. Werner Raab (approximately 2ooo hrs) and by Mr. Walter Majerotto (approximately 2oo hrs). Furthermore, some 2oo - 25o hrs have been spent by translators, typist, secretariate, bookkeeping, correspondence.

The following 4 Special Reports have been produced:

Nr. 1 "On the theory of similitude of matrices".	15.February 1963
Nr. 2 "On a generalisation of the spinor concept to Riemannian spaces".	7. June 1963
Nr. 3 "On the relationship between tensors and spinors and the conversion of indices in Riemannian spaces".	2o. July 1963
Nr. 4 "The decomposition of the Gordon operator".	25. July 1963

An abbreviated form of this final report will be submitted shortly to Acta Physica Austriaca for publication.

Expenses (in Austria Scillings):

	Paid	Due, but not yet paid	Total
Scientific Personnel	7o.366,7o	3.944,06	74.31o,76
Administrative personnel	3.663,2o	1.ooo,--	4.663,2o
Property Acquired	792,15	--	792,15
All Other Expenses	4.567,86	--	4.567,86
	79.389,91	4.944,o6	84.333,97

Acquired property:

Scientific Literature: 3 books.

Received funds:

First, second and third report received.

Final report appr. A.S. 21.1o3,--

Total received funds will be appr. A.S. 84.3oo,--